W10 Hilbert's Nullstellensatz

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closure in Zariski topology.

Lemma 7.1 If $X \subseteq \mathbb{A}^n$, then $V(I(X)) = \overline{X}$

In particular: $V(I(X)) = X$ for X an affine variety.

Proof: $X \subseteq V(I(X))$ and $V(I(X))$ is closed, so $\overline{X} \subseteq V(I(X))$.

"$\subseteq$": $\overline{X} = \bigcap_{\substack{f \in K[x_1,\dots,x_n] \\ X \subseteq V(f)}} V(f)$. Let f be s.t. $X \subseteq V(f) \Rightarrow f \in I(X) \Rightarrow V(I(X)) \subseteq V(f)$

$\Rightarrow V(I(X)) \subseteq \overline{X}$. $\square$

Thm 7.2 (Hilbert's Nullstellensatz, version 2/ strong form) Let K be algebraically closed.

Then, for $J \trianglelefteq K[x_1,\dots,x_n]$: $I(V(J)) = \sqrt{J}$.

In particular, there is an inclusion-reversing bijection

An ideal J is radical if $J = \sqrt{J}$

$$\{\text{affine varieties in } \mathbb{A}^n(K)\} \longleftrightarrow \{\text{radical ideals of } K[x_1,\dots,x_n]\}$$
$$X \longmapsto I(X)$$
$$V(J) \longleftarrow J$$

Thm 7.3 (HNS, v.3) Let K be alg. closed, $R = K[x_1,\dots,x_n]$.

Then $\mathrm{Max}(R) = \{(x_1 - a_1,\dots, x_n - a_n) : a_1,\dots,a_n \in K\}$

Proof: "$\supseteq$" ✓, "$\subseteq$" Let $M \in \mathrm{Max}(R)$, R/M is a field, $\varphi: R \to R/M$

has $\ker \varphi \cap K = 0$, so $K \cong \varphi(K)$. So R/M is a field ext. of the alg. closed $\varphi(K)$, but also $R/M = \varphi(K)[\varphi(x_1),\dots,\varphi(x_n)]$ is a f.g. $\varphi(K)$-algebra $\overset{T6.16}{\Longrightarrow} \dim_{\varphi(K)} R/M < \infty \overset{\varphi(K) \text{ alg. closed}}{\Longrightarrow} R/M = \varphi(K)$.

Let $a_1,\dots,a_n \in K$ s.t. $\varphi(x_i) = \varphi(a_i) \Rightarrow (x_1 - a_1,\dots, x_n - a_n) \subseteq \mathrm{Ker}(\varphi) = M$

$\Rightarrow M = (x_1 - a_1,\dots, x_n - a_n)$ by maximality. $\square$

I.e., if K alg. closed, there is a bijection

$$\mathrm{Max}(K[x_1,\dots,x_n]) \leftrightarrow \mathbb{A}^n(K).$$

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For $I \trianglelefteq R$, $\sqrt{I} = \bigcap_{\substack{M \in \mathrm{Max}(R) \\ I \subseteq M}} M$.

Proof: „$\subseteq$" $\sqrt{I} = \bigcap_{\substack{P \in \mathrm{Spec}(R) \\ I \subseteq P}} P$, and $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$

„$\supseteq$" Let $f \in R \setminus \sqrt{I}$. Show: $\exists M \in \mathrm{Max}(R)$: $M \supseteq \sqrt{I}$, $f \notin M$.

Let $S = \{f^n : n \geq 0\} \Rightarrow S \cap \sqrt{I} = \emptyset \Rightarrow S^{-1}\sqrt{I} \underset{\neq}{\trianglelefteq} S^{-1}R = R_f$

Let $M' \in \mathrm{Max}(R_f)$, $M' \supseteq S^{-1}\sqrt{I}$. Let $M := M' \cap R \in \mathrm{Spec}(R)$

$\Rightarrow M \supseteq \sqrt{I}$, $f \notin M$. Show: $M \in \mathrm{Max}(R)$.

Here: $K \hookrightarrow R/M \hookrightarrow (R/M)_{\bar{f}} = R_f/M_f$ (since $\bar{f} := f+M \neq \bar{0}$ in the domain R/M)

$\uparrow$ R/M domain

$M_f = M' \in \mathrm{Max}(R_f)$, so R_f/M_f is a field, f.g. as K-algebra (by images of gens of R and $\frac{1}{f}$).

$\overset{T6.16}{\Longrightarrow} [R_f/M_f : K] < \infty \Rightarrow R/M$ is a fin. dim. domain $/K$.

$\Rightarrow R/M$ is a field.

[Alternatively: R_f/M_f integral $/K \Rightarrow R/M$ integral $/K \Rightarrow \dim(R/M) = \dim(K) = 0$.]

$\square$

Proof of Thm 7.2: Let $R = K[x_1, \dots, x_n]$, $J \trianglelefteq R$.

$I(V(J)) \supseteq J$, so also $I(V(J)) \supseteq \sqrt{J}$.

„$\subseteq$": Suppose $f \notin \sqrt{J}$. Show: $f \notin I(V(J))$

$\overset{T7.4}{\Longrightarrow} \exists M \in \mathrm{Max}(R)$: $\sqrt{J} \subseteq M$ but $f \notin M$.

By T7.3, $M = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in K$.

Consider $\varphi: R \to R/M$, $x_i \mapsto a_i$. Then $\varphi(g) = g(a_1, \dots, a_n)$ $\forall g \in R$.

$$\left.\begin{array}{l} \text{Now } f \notin M \Rightarrow f(a_1, \dots, a_n) \neq 0. \\ \text{But } M \supseteq \sqrt{J} \Rightarrow (a_1, \dots, a_n) \in V(M) \subseteq V(J) \end{array}\right\} \Rightarrow f \notin I(V(J))$$

Correspondence holds by L7.1.

$\square$

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Cor 7.5 (HNS, weak ver.) If K is alg. closed, $I \subsetneq K[x_1,\ldots,x_n]$
$\Rightarrow V(I) \neq \emptyset$

[Bec. $1 \notin I \Rightarrow 1 \notin \sqrt{I}$, so $\sqrt{I}$ is proper. Apply Thm 7.2]

Contrapositive: If $f_1,\ldots,f_m \in K[x_1,\ldots,x_n]$ do not have a common zero,
$\exists g_1,\ldots,g_m \in K[x_1,\ldots,x_n]$: $\quad 1 = f_1 g_1 + \ldots + f_m g_m$.

Def: K field. For a variety $X \subseteq \mathbb{A}^n(K)$, the (affine) coordinate ring is $A(X) := K[x_1,\ldots,x_n]/I(X)$.

Cor 7.6 Let K be alg. closed, $X \subseteq \mathbb{A}^n$ a variety. Then there is an inclusion-reversing bijection

$$\begin{array}{ccccc}
\{\text{subvarieties of } X\} & \longleftrightarrow & \{\text{radical ideals of } A(X)\} & & \\
Y & \longmapsto & I(Y)/I(X) & & \\
V(J) & \longleftarrow & J/I(X) & & (J \trianglelefteq K[x_1,\ldots,x_n],\ J \supseteq I(X))
\end{array}$$

Points on X map to $\mathrm{Max}(A(X))$.

Proof: By T7.2, there is a bij. $(*)$

$$\{\text{subvarieties of } X\} \overset{(*)}{\longleftrightarrow} \left\{\begin{array}{l}\text{radical ideals } J \trianglelefteq K[x_1,\ldots,x_n] \\ \text{with } I(X) \subseteq J\end{array}\right\}$$

$$\longleftrightarrow \{\text{radical ideals of } A(X)\}$$ $\square$